Lecture XV Measurements are matrices ; States are vectors

① In lecture XIII we noted that in quantum mechanics we divide the world into observers (who make measurements) & systems (which are measured). In Lecture XIV we found (a) that the measurement of polarization does not commute and square matrices do not commute

Today we combine these ideas. We make measurements into square matrices. What should we make systems to be?

② The following is a hint:

$$(\text{Measurement})(\text{system}) = \text{system}$$
$$(\text{Square Matrix})(\text{system}) = \text{system}$$

What mathematical object which when subjected to a measurement (a square matrix) comes back as a mathematically similar object?

Answer: Column vectors.

$$\begin{pmatrix}\text{Square} \\ \text{Matrix}\end{pmatrix}(\text{col. vector}) = \text{col. vector}.$$

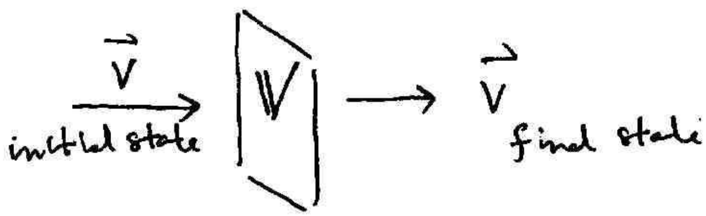
③ ∴ Measurement $\rightarrow$ Square Matrix, M

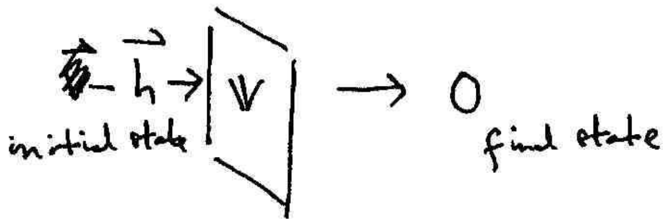
System (or state) $\rightarrow$ Column vector, $\vec{V}$

& if $\vec{V}$ is a wave state $\vec{V}^\dagger \vec{V}$ is a number. Let's say, this number is probability, the intensity of $\vec{V}$

④ If we call vertical polarized light, $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
and " " horizontal " " , $\vec{h} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

⑤ We would like to express:





gets written as $\mathbb{V}\,\vec{v} = \vec{v}$

$$\mathbb{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\& \quad \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and similarly $\mathbb{V}\,\vec{h} = 0$

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

⑥ We can find the value of $\mathbb{V}$.

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\therefore \quad V_{11} = 1$$
$$V_{21} = 0$$

and
$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad V_{12} = 0$$
$$V_{22} = 0$$

Conclusion $\mathbb{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

⑦ In the same way we can consider diagonal right polarized light $\vec{d}_R$.

$$\vec{d}_R = \begin{pmatrix} d_{R1} \\ d_{R2} \end{pmatrix}$$

What are the values of d_{R1} and d_{R2}?

⑧ Note
$$\begin{pmatrix} d_{R1} \\ d_{R2} \end{pmatrix} = d_{R1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d_{R2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= d_{R1}\,\vec{v} + d_{R2}\,\vec{h}$$

As diagonal polarized light is half-way inbetween $\vec{v}$ & $\vec{h}$ light, we expect by symmetry

$$d_{R1} = d_{R2}$$

We conclude $\vec{d}_R = \begin{pmatrix} d_{R_1} \\ d_{R_1} \end{pmatrix}$.

⑨ What is the value of d_{R1}? To answer this recall if $\vec{d}_R$ is the wave-state,

The intensity is prob. amplitude times its complex conjugate. Translating to the state vectors

$$(\text{row vector})^* (\text{column vector}) = + \text{ real number}.$$

$\therefore$ the ~~probability~~ or intensity of $\vec{d}_R$ state is

$$\vec{d}_R^{\,\dagger} \vec{d}_R = (d_{R1} \ d_{R1}) \begin{pmatrix} d_{R1} \\ d_{R1} \end{pmatrix} = 2 d_{R1}^2$$

⑩ We would like the intensity of $\vec{d}_R$ light to be the same as $\vec{v}$ light

$$\vec{v}^{\dagger} \vec{v} = (1 \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\therefore \quad 2 d_{R1}^2 = 1 \quad \& \quad d_{R1} = \sqrt{1/2} = \frac{1}{\sqrt{2}}$$

$$\& \quad \vec{d}_R = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

⑪ Review

Measurements are expressed a matrices; $\mathbb{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Systems are expressed as vectors; $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{h} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\vec{d}_R = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

, Intensities are for a vector $\vec{w}$: $\vec{w}^{\dagger} \vec{w}$.

The intensities of $\vec{v}$, $\vec{h}$ & $\vec{d}_R$ are all equal to 1 as

$$(1 \ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1, \quad (0 \ 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \quad \& \quad (1/\sqrt{2} \ 1/\sqrt{2})\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 1$$

Excercise 1. In the same way we found $\mathbb{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\vec{h} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ find the matrix $\mathbb{H}$, which corresponds to a horizontal polarizing screen.