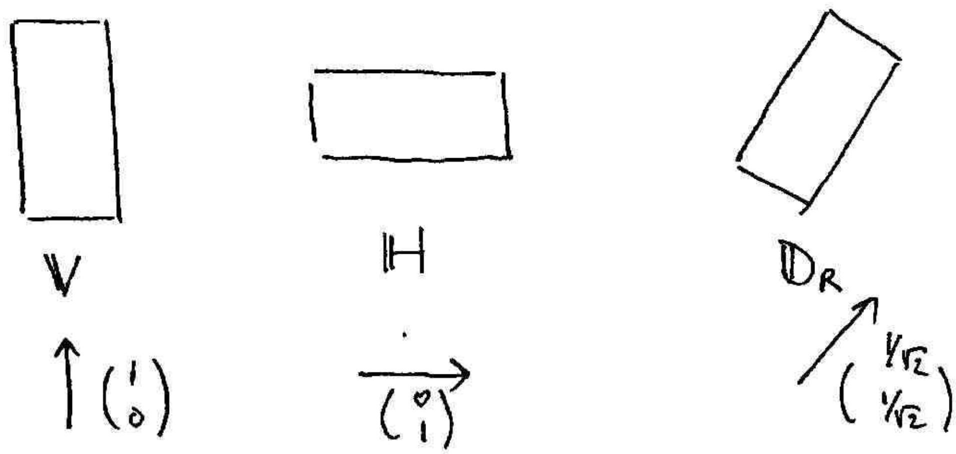
Lecture $\underline{\overline{XVI}}$ Measurements, states, matrices & vectors

① In the last lecture we found measurements could be expressed as matrices and states as vectors

$$\mathbb{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{H} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{h} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \& \quad \vec{d}_R = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$



② Let's find the matrix which corresponds to $\mathbb{D}_R$

To do so we note the following

$$\mathbb{D}_R \vec{d}_R = \vec{d}_R \quad \therefore \quad \begin{pmatrix} D_{R11} & D_{R12} \\ D_{R21} & D_{R22} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore (D_{R11})\frac{1}{\sqrt{2}} + (D_{R12})\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$(D_{R21})\frac{1}{\sqrt{2}} + (D_{R22})\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad D_{R11} + D_{R12} = 1$$

$$D_{R21} + D_{R22} = 1$$

③ $\mathbb{D}_R \vec{v}$ will turn some $\vec{v}$ light into $\vec{d}_R$ light

$\mathbb{D}_R \vec{h}$ " " " $\vec{h}$ " " " "

As $\mathbb{D}_R$ is exactly half way between $\mathbb{V}$ & $\mathbb{H}$.

$$\mathbb{D}_R \vec{v} = \mathbb{D}_R \vec{h}$$

$$\therefore \begin{pmatrix} D_{R11} & D_{R12} \\ D_{R21} & D_{R22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} D_{R11} & D_{R12} \\ D_{R21} & D_{R22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} D_{R11} \\ D_{R21} \end{pmatrix} = \begin{pmatrix} D_{R12} \\ D_{R22} \end{pmatrix}$$

As $D_{R11} + D_{R12} = 1$ & $D_{R11} = D_{R12}$

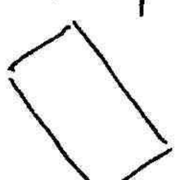
$$\therefore D_{R11} = D_{R12} = 1/2$$

Similarly $D_{R21} + D_{R22} = 1$ & $D_{R21} = D_{R22}$

$$\therefore D_{R21} = D_{R22} = 1/2$$

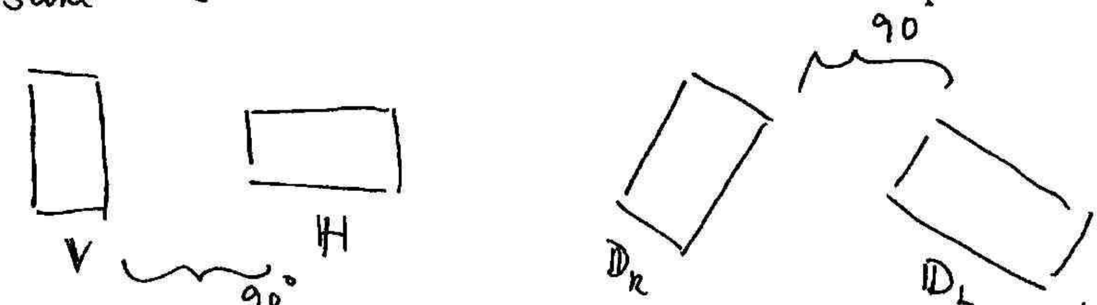
$$\mathbb{D}_R = \begin{pmatrix} D_{R11} & D_{R12} \\ D_{R21} & D_{R22} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

④ We can now find $\vec{d}_L$ & $\mathbb{D}_L$ where $\mathbb{D}_L$ is a left-diagonal polarizing screen



$\mathbb{D}_L$.

⑤ To find $\mathbb{D}_L$ & $\vec{d}_L$ we note that the relatin between $\mathbb{V}$ and $\mathbb{H}$ is the same as that between $\mathbb{D}_R$ and $\mathbb{D}_L$.



Both are rotated 90° with respect to the other.

Note $\quad \mathbb{V}\vec{h} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ $\vec{d}_L$ light is light such that

$$\mathbb{D}_R \vec{d}_L = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} d_{L1} \\ d_{L2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We see $\frac{1}{2} d_{L1} + \frac{1}{2} d_{L2} = 0$ & $\therefore d_{L1} = -d_{L2}$

By intensity we require $(d_{L1} \; d_{L2})\begin{pmatrix} d_{L1} \\ d_{L2} \end{pmatrix} = 1$

$\therefore \; d_{L1}^2 + d_{L2}^2 = 1 \quad \therefore d_{L1}^2 + d_{L1}^2 = 1$

& $d_{L1}^2 = \frac{1}{2}$ $\quad d_{L1} = 1/\sqrt{2}$. $\quad d_{L2} = \frac{1}{\sqrt{2}}$

Result: $\quad d_L = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

Excercise 2 $\quad$ Show $\mathbb{D}_L = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

⑥

We have now found the values of $\mathbb{V}$, $\mathbb{H}$, $\mathbb{D}_R$, $\mathbb{D}_L$, $\vec{v}$, $\vec{h}$, $\vec{d}_R$ & $\vec{d}_L$.

$$\mathbb{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbb{H} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbb{D}_R = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad \mathbb{D}_L = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{h} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \vec{d}_R = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \vec{d}_L = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

We can use these equations to calculate the following

$\vec{v} \rightarrow \boxed{\mathbb{V}} \rightarrow \,?$ $\qquad \mathbb{V}\vec{v} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Int. $= 1$

$\vec{h} \rightarrow \boxed{\mathbb{V}} \rightarrow \,?$ $\qquad \mathbb{V}\vec{h} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Int. $= 0$

$\vec{d}_R \rightarrow \boxed{\mathbb{V}} \rightarrow \,?$ $\qquad \mathbb{V}\vec{d}_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}$ Int. $= 1/2$

$\vec{d}_R \rightarrow \boxed{\mathbb{V}} \rightarrow \boxed{\mathbb{D}_R} \rightarrow$ $\qquad \mathbb{D}_R \mathbb{V} \vec{d}_R = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{pmatrix}$

Int. $= 1/4$

Nomenclature $\mathbb{M}\vec{w} = t\vec{w}$ $\quad \vec{w}$ is an eigenvector of $\mathbb{M}$ & t is the eigenvalue.

$\mathbb{V}$ has two eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eigenvalue 1

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ " " 0

<u>so</u>
$$\mathbb{V}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\mathbb{V}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note : After a measurement such as $\mathbb{V}$, the system is in an eigenvector of $\mathbb{V}$.

$$\mathbb{V}\, \vec{d}_R = \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\& \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}$$

This is always true in quantum mechanics.