Extra Lecture (E17)

1) Last time we ended the lecture with the assertion that any state can be written as a linear combination of the eigenvectors of a measurement, $\mathbb{M}$ (the completeness relation)

Example

$$\vec{d}_L = \frac{1}{\sqrt{2}}\vec{v} - \frac{1}{\sqrt{2}}\vec{h}$$

$\vec{v}$ and $\vec{h}$ are the eigenvectors of $\mathbb{V}$.

2) The coefficients of the vectors are the Probability Amplitudes. The P.A. that $\vec{d}_L$ behaves like $\vec{v}$ is $\frac{1}{\sqrt{2}}$.
" " " " $\vec{h}$ " $-\frac{1}{\sqrt{2}}$.

3) We stated a rule for all $\vec{w}^{\dagger}\vec{w} = 1$

$$\vec{w}^{\dagger}\vec{u}^{*}$$

is the PA that $\vec{u}$ behaves like $\vec{w}$

In today's lecture we are going to find out some of the consequences of these statements.

Let us consider the measurement of the energy, $\mathbb{H}$ (the letter $\mathbb{H}$ is always used here.).

Let $\vec{\psi}_1$ and $\vec{\psi}_2$ be the only eigenvectors of $\mathbb{H}$ and let $\vec{\phi}_a$ and $\vec{\phi}_b$ be two states which are not eigenvectors. Let E_1 and E_2 be the eigenvalues of respectively $\vec{\psi}_1$ and $\vec{\psi}_2$ i.e.,

$$\mathbb{H}\,\vec{\psi}_1 = E_1 \vec{\psi}_1$$

$$\mathbb{H}\,\vec{\psi}_2 = E_2 \vec{\psi}_2$$

Assume $\vec{\psi}_j^{\dagger} \psi_j = 1$

Assume $\vec{\phi}_j^{\dagger} \vec{\phi}_j = 1$

5) As any state can be written as a linear combination of $\vec{\psi}_1$ and $\vec{\psi}_2$

$$\vec{\phi}_a = a_1 \vec{\psi}_1 + a_2 \vec{\psi}_2$$

$$\vec{\phi}_b = b_1 \vec{\psi}_1 + b_2 \vec{\psi}_2$$

6) The coefficients a_1, a_2, b_1 and b_2 are P.A.

$$a_1 = \vec{\psi}_1^{\dagger} \vec{\phi}_a \qquad a_2 = \vec{\psi}_2^{\dagger} \phi_a \qquad b_1 = \vec{\psi}_1^{\dagger} \vec{\psi}_b \qquad b_2 = \vec{\psi}_2^{\dagger} \psi_b$$

7) & the Prob. for example $\vec{\phi}_a$ behave like $\vec{\psi}_1$ is

$$\text{Prob} = a_1^* a$$

⑦ Thm. If $\vec{\psi}_1$ and $\vec{\psi}_2$ are eigenvectors with different eigenvalues:

$$\vec{\psi}_1^{\,\dagger} \vec{\psi}_2 = 0$$

Pf. $\vec{\psi}_2 = 1\,\vec{\psi}_2 + 0\,\vec{\psi}_1$

The P.A. that $\vec{\psi}_2$ behaves like $\vec{\psi}_2$ is 100%

" " " " " $\vec{\psi}_1$ " 0%

But the P.A. (by our definition) is

P.A. that $\vec{\psi}_2$ behaves like $\vec{\psi}_1$ $= \vec{\psi}_1^{\,\dagger} \vec{\psi}_2$ QED

⑧ Thm $\vec{\phi}_a^{\,\dagger} \mathbb{H} \vec{\phi}_a$ = average energy measured for the $\vec{\phi}_a$ state.

(Recall $\vec{\phi}_a^{\,\dagger} \vec{\phi}_a = 1$)

a_1 is PA $\vec{\phi}_a$ behaves like $\vec{\psi}_1$
a_2 " " $\vec{\phi}_a$ " " $\vec{\psi}_2$

Pf. $\vec{\phi}_a^{\,\dagger} \mathbb{H} \vec{\phi}_a = \left(a_1^* \vec{\psi}_1^{\,\dagger} + a_2^* \vec{\psi}_2^{\,\dagger}\right) \mathbb{H} \left(a_1 \vec{\psi}_1 + a_2 \vec{\psi}_2\right)$

$= \left(a_1^* \vec{\psi}_1^{\,\dagger} + a_2^* \vec{\psi}_2^{\,\dagger}\right)\left(a_1 E_1 \vec{\psi}_1 + a_2 E_2 \vec{\psi}_2\right)$

$= a_1^* a_1 E_1 \vec{\psi}_1^{\,\dagger}\vec{\psi}_1 + a_2^* a_1 E_1 \vec{\psi}_2^{\,\dagger}\vec{\psi}_1 + a_1^* a_2 E_2 \vec{\psi}_1^{\,\dagger}\vec{\psi}_2 + a_2^* a_2 E_2 \vec{\psi}_2^{\,\dagger}\vec{\psi}_2$

$= a_1^* a_1 E_1 \quad + \quad 0 \quad + \quad 0 \quad + a_2^* a_2 E_2$

$= a_1^* a_1 E_1 + a_2^* a_2 E_2$

ut what is the meaning in words of

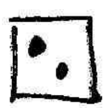
$$a_1^* a_1 E_1 + a_2^* a_2 E_2 \ ?$$

$a_1^* a_1$ is the Prob. that $\vec{\phi}_a$ behaves like $\vec{\psi}_1$.

$a_2^* a_2$ " " " " " " $\vec{\psi}_2$.

ince $\vec{\phi}_a$ either behaves like $\vec{\psi}_1$ or $\vec{\psi}_2$
it is a lin. comb. of $\vec{\psi}_1$ and $\vec{\psi}_2$) this then
the average energy measured for the $\vec{\phi}_a$ state.
QED

This little example may help.

Consider a crooked die. 2/3 of the time it produces . 1/3 of the time it produces . What is the average value of a throw?

Prob. value Prob. value

Answer: $\frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 5 = 3$

This is analogous to the above example.

(10) This rule is always true in quantum mechanics. If $\mathbb{M}$ is a measurement and $\vec{\phi}$ is a state where $\vec{\phi}^{\dagger}\vec{\phi} = 1$

$$\vec{\phi}^{\dagger}\,\mathbb{M}\,\vec{\phi} = \text{average measured value of } \mathbb{M} \text{ for } \vec{\phi}.$$

(11) One last point: We can also give a meaning to:

$$\vec{\phi}_b^{\dagger}\,\mathbb{H}\,\vec{\phi}_a$$

$\mathbb{H}\vec{\phi}_a$ is what happens to the state $\vec{\phi}_a$ after measurement of the energy. Let's call

$$\mathbb{H}\vec{\phi}_a = \vec{\chi}$$

then $\vec{\phi}_b^{\dagger}\,\mathbb{H}\,\vec{\phi}_a = \vec{\phi}_b^{\dagger}\,\vec{\chi}$ where $\vec{\phi}_b^{\dagger}\vec{\phi}_b = 1$.

$\therefore\ \vec{\phi}_b\,\mathbb{H}\,\vec{\phi}_a$ is the P.A. that $\vec{\chi}$ (or $\mathbb{H}\vec{\phi}_a$) behaves like $\vec{\phi}_b$.